Postscript to Note on a Point in Vulgar Fractions.

By J. J. Sylvester.

Let ϕx represent $x^2 - x + 1$, $\phi^n c$ will then be the general term of the "limiting sorites" whose first term is c, for which, if we please, $1 - c$ may be substituted. The properties of the numbers $\phi^n c$ seem to be worthy of some attention. I confine my observations in what follows to the lowest of such series, viz. where $c = 2$ or -1.

The first five terms in such series then become $\bar{1}$ or 2, 3, 7, 43, 1807, 3263443, of which all but 1807, which = 13.139, are prime numbers. Every term in the series must contain only factors of the form $6i + 1$, and this, joined to the fact that a prime factor which has once appeared in any term can never reppear in any other, favors a tendency, so to say, of the numbers to remain primes, or at all events, to be of very limited frangibility into a product of primes.

It is easy to determine whether any proposed prime can occur as a factor of any term whatever in the series; for taking that number, say p, as a modulus, if r is a remainder of any term to that modulus, the remainder of the next term will be $r^2 - r + 1$, and as soon as any remainder reappears the series of remainders becomes periodic; so that necessarily in less than the number p of remainders, if p does divide any term of the sorites, we must arrive at a remainder zero, subsequent to which all the remainders are unity. I give the remainders and periods in the annexed table for all values of p of the form $6i + 1$ up to 139, from which it will be seen that, under that limit, 13 and 73 are the only prime numbers which are contained as factors in the terms of the series.

p	Remainders of $\phi^n(2)$ to modulus p.
2	0.
3	2, 0.
7	2, 3, 0.
13	2, 3, 7, 4, 0.
19	2, 3, 7, 5; 2, 3, 7, 5; . . . .
31	2, 3, 7, 12, 9, 11, 18, 28, 13; 2, 3, 7, ... , 13; . . . .
37	2, 3, 7; 6, 31; 6, 31; . . . .
43	2, 3, 7, 0.
61	2, 3, 7, 43, 38, 4, 13, 35, 32; 17, 29, 20, 15, 28, 25, 52, 30; . . . .
67	2, 3; 7, 43, 65; 7, 43, 65; . . . .
73	2, 3, 7, 43, 55, 51, 69, 21, 56, 15, 65, 0.
79	2, 3, 7; 43, 69, 32, 45, 6, 31, 61, 27, 71, 73; . . . .
97	2, 3, 7, 43, 61; 72, 69, 37; 72, 69, 37; . . . .
103	2, 3; 7, 43, 56, 94, 91, 54, 82, 51, 79, 86, 101; 7, 43, ...; . . . .
109	2, 3, 7, 43, 63, 92, 89, 94, 23, 71, 66, 40, 35, 101, 73, 25, 56, 29, 50, 53; 32, 12, 24, 8; 32, 12, 24, 8; . . . .
127	2, 3, 7, 43, 29, 51, 11; 111, 19, 89, 86, 72, 33, 41, 117; . . . .
139	2, 3, 7, 43, 0.
151	2, 3, 7, 43, 146; 31, 25, 148, 13, 6; . . . .
157	2, 3; 7, 43, 80, 41, 71, 104, 37, 77, 44, 9, 73, 76, 49, 155; . . . .
163	2, 3; 7, 43, 14, 20, 55, 37, 29, 161; . . . .
181	2, 3, 7, 43, 178, 13, 157, 58, 49, 0.
193	2; 3, 7, 43, 70, 6, 31, 159, 33, 92, 74, 192; . . . .
199	2, 3; 7, 43, 16, 42, 131, 116, 8, 57, 9, 73, 83, 41, 49, 164, 67, 45, 190, 91, 32, 197; . . . .